First Steps in Non-commutative Geometry

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Updated 2017 Under construction...

Contents

Introduction

In this course we hope to introduce some of the basic ideas and applications behind the reasonable new field of mathematics of non-commutative geometry. We aim to do this in a way so that undergraduates can approach the subject without feeling overwhelmed by the scope of the subject.

1 Lecture 1

1.1 Topology

Throughout the semester we will add the necessary topological definitions required in order to prove the required results. For the interested reader, they are refered to cite for a more detailed introduction to the topic of topology

Definition 1.1: A topology

Given a set X, a *topology*, τ on X is a collection of subsets of X such that the following rules hold:

(i) The empty set, $\emptyset \in \tau$, and $X \in \tau$.

(ii) If $U_1, U_2 \in \tau$ then $U_1 \cap U_2 \in \tau$

(iii) If $\{U_i \in \tau; i \in I\}$, where I can be any indexing set, then $\bigcup_{i \in I} U_i \in \tau$

We call the pair (X, τ) a *topological space*, however for the sake of brevity we often denote a topological space as simply X. The elements of X are called the points of the space. The elements of τ are the *open sets* of the space and are said to be *open* in (X, τ).

Remark 1. It can be seen via induction that any finite intersection of open sets is also open. Also, notice that in the third condition, the indexing set can be infinite or finite, in fact, it can even be uncountably infinite.

Example 1. *Let X be a set. The set $\tau = 2^X$, i.e. the set of all subsets of X, often referred to as the power set, is a topology on X. This topology is referred to as the discrete topology.*

Example 2. *Let X be a set. The set $\tau = \{\emptyset, X\}$ is a topology on X and is referred to as the trivial topology.*

Definition 1.2: Continuity

Let X and Y be topological spaces. We say a function $f : X \to Y$ is *continuous* if for every open set $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open in X.

Remark 2. Note that the pre-image $f^{-1}(U) = \{x \in X; f(x) \in U\}$ is well defined regardless of whether f is invertible or not.

Example 3. *Let $f : X \to Y$ where X is a topological space with discrete topology, and Y is an arbitrary topological space. Then the all functions must be continuous.*

1.2 Algebra

Definition 1.3: An algebra

An *algebra* $\mathcal{A}$ is a vector space $\mathcal{V}$, over a field $\mathbb{F}$, with a bilinear product $(a, b) \in \mathcal{V} \times \mathcal{V} \mapsto ab \in \mathcal{V}$ such that:

$$(ab)c = a(bc), \quad \forall a, b, c \in \mathcal{A} \quad (Associativity)$$

We say that the algebra is *unital* if there exists $1 \in \mathcal{A}$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathcal{A}$. We also say that an algebra is *commutative* if $ab = ba, \forall a, b \in \mathcal{V}$ and *non-commutative* if this condition is not satisfied.

Remark 3. Actually an algebra can also be non-associative, but we are not concerned with this case for the purpose of our work.

Example 4. *Let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries. This space is a unital non-commutative algebra with algebra operators as the usual matrix operations.*

Definition 1.4: Commutativity

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. An *involution* is a linear map $* : \mathcal{A} \to \mathcal{A}$ such that:

(i) $(a^*)^* = a \ , \quad \forall a \in \mathcal{A}$

(ii) $(ab)^* = b^* a^*, \quad \forall a, b \in \mathcal{A}$

(iii) $(\alpha a)^* = \bar{\alpha} a^*, \quad \alpha \in \mathbb{C}, a \in \mathcal{A}$

We refer to an algebra with an involution as a $*$-*algebra*.

Definition 1.5: Algebra homomorphism

Let $\mathcal{A}$ and $\mathcal{B}$ be $*$-algebras. An *algebra $*$-homomorphism* between $\star$-algebras $\mathcal{A}$ and $\mathcal{B}$, is a linear map $\phi : \mathcal{A} \to \mathcal{B}$ such that:

(i) $\phi(ab) = \phi(a)\phi(b)$

(ii) $\phi(a^\star) = \phi(a)^\star$

Remark 4. Notice here that we explicitly said algebra $*$-homomorphism. We can also have group homomorphisms, algebra homomorphisms and many more. A homomorphism can be thought of as a structure preserving map, with respect to the structure found on the space.

Example 5. *$M_n(\mathbb{C})$ with operations defined above, and involution defined as complex conjugate transpose, is a $*$-algebra.*

Definition 1.6: Algebra isomorphism

Let ϕ be an algebra $*$-homomorphism between $*$-algebras $\mathcal{A}$ and $\mathcal{B}$. We say that $\mathcal{A}$ is isomorphic to $\mathcal{B}$, denoted by $\mathcal{A} \simeq \mathcal{B}$, if ϕ is invertible.

Remark 5. Once again, the notion of isomorphism extends to many mathematical structures. It is simply and invertible isomorphism.

Definition 1.7: Matrix algebra

We say A is a complex matrix algebra if:

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}), \quad n_i \in \mathbb{N}$$

with involution defined as conjugation transpose and the algebra structures defined as the usual product and additions on matrices.

1.3 Finite Commutative Geometry

In order to make sense of ”non-commutativity” in geometry, we require a correlation between algebras and geometry. For now, we will work within the framework of finite geometries/algebras and extend to the full theory later.

Proposition 1.1

Let X be a finite topological space with discrete topology. Then the space of continuous functions, $f : X \to \mathbb{C}$, labelled by $C(X)$, is a commutative matrix algebra, up to isomorphism, with products defined via the following relations:

(i) $(\lambda f + g)(x) = \lambda f(x) + g(x), \quad \forall f, g \in C(X), \forall x \in X, \forall \lambda \in \mathbb{C}$

(ii) $(fg)(x) = f(x)g(x), \quad \forall f, g \in C(X), \forall x \in X$

(iii) $f^*(x) = \overline{f(x)}, \quad \forall f, g \in C(X), \forall x \in X$

Proof. As X has discrete topology, it is clear that all functions from X to $\mathbb{C}$ are continuous. Now, define $\phi : C(X) \to \mathbb{C}^N$ as:

$$\phi(f) = \begin{pmatrix} f(1) \\ f(2) \\ \vdots \\ f(N) \end{pmatrix}$$

where $i \in \{1, \ldots, N\}$ labels the points in X and $N = |X| \in \mathbb{N}$. It is clear that ϕ is a homomorphism between $C(X)$ and $\mathbb{C}^N$. To show it is an isomorphism, we must simply show the ϕ is both injective and surjective. For injective, suppose that $\phi(f) = \phi(g)$ with $f, g \in C(X)$. Then, $f(i) = g(i)$ for all $i \in \{1, \ldots, N\}$. Thus we have that $f = g$. For surjective, take:

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

As all functions are continuous, pick the function f such that $f(i) = v_i$ for each $i \in \{1, \ldots, N\}$. So, $C(X) \simeq \mathbb{C}^N \simeq \mathbb{C} \underbrace{\oplus \cdots \oplus}_{N \text{ times}} \mathbb{C}$. Thus $C(X)$ is isomorphic to a commutative matrix algebra. □

This is an important result. We have shown that given a finite topological space, we can construct an algebra ($*$-algebra), unique up to the cardinality of X. Shown figuratively,

$$\text{Finite Topological Spaces} \longrightarrow \text{Commutative Matrix Algebras}$$

Unfortunately, this is relatively useless in generalising the notions introduced in differential geometry/topology. In order to do this, we must construct a reverse arrow. In other words, can one construct a finite topological space, X, from an arbitrary matrix algebra, A, where $A \simeq C(X)$. This is obviously not true as matrix algebras are generally not commutative and $C(X)$ is commutative. There are two ways of resolving this:
(1)Restrict ourselves to commutative (diagonal) matrix algebras.
(2)Allow a new notion of "isomorphism" so that A can be "isomorphic" to $C(X)$.
Out of these two options, option (2) is the most interesting to lead to an extension of ordinary topology. However, laying out the methods needed in option (1) will give us a starting point in order to understand the extension.

1.4 Exercises

Exercise 1: *Show that $\phi : X_1 \to X_2$ is an injective(surjective) map of finite spaces if and only if $\phi^* : C(X_2) \to C(X_1)$ is surjective(injective).*

Exercise 2: *Explain the implications of the above exercise with respect to todays lecture. Hint: Think about what this implies with regards to uniqueness in proposition 1.15.*

2 Lecture 2

Aim of lecture is to construct the following reverse arrow:

$$\text{Finite Topological Spaces} \longleftarrow \text{Commutative Matrix Algebras}$$

In order to do this, we will require a few technical definitions.

2.1 Finite Hilbert Spaces and Linear Operators

Definition 2.1: Inner product

Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. An *inner product* on $\mathcal{V}$ is a map $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ such that:

(i) $\langle v, u \rangle = \overline{\langle u, v \rangle}$ for $v, u \in \mathcal{V}$

(ii) $\langle \alpha v, u \rangle = \alpha \langle v, u \rangle$ and $\langle v + u, w \rangle = \langle v, w \rangle + \langle u, w \rangle$ for $\alpha \in \mathbb{F}, u, v, w \in \mathcal{V}$

(iii) $\langle v, v \rangle \geq 0,\ v \in \mathcal{V}$ and $\langle v, v \rangle = 0$ iff $v = 0$

We refer to the pair $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ as a *inner product space*.

Definition 2.2: Finite dimensional inner product space

Let $\mathcal{H}$ be a finite dimensional inner product space. We will refer to this space as a *finite Hilbert space*.

Remark 6. Strictly speaking, the above definition requires a proof. For now, we will not prove this statement as it is not completely enlightening to this part of the course. We will however prove this result later when working with general infinite dimensional vector spaces and have a more rigorous definition of a Hilbert Space.

Definition 2.3: An operator

Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over a field $\mathbb{F}$. We say an operator $L : \mathcal{U} \to \mathcal{V}$, is *linear* if

$$L(\alpha u + u') = \alpha L(u) + L(u') \quad \forall u, u' \in \mathcal{U}, \alpha \in \mathbb{F}$$

Remark 7. We denote the space of linear maps from $\mathcal{U}$ to $\mathcal{V}$ by $L(\mathcal{U}, \mathcal{V})$. If we write $L(\mathcal{U})$ it is assumed to mean $L(\mathcal{U},\mathcal{U})$.

Remark 8. Note that linear maps are just the special case of homomorphisms when applied to vector spaces.

2.2 Necessary Proofs

Proposition 2.1

Let A be a commutative matrix algebra. Then $A \simeq \mathbb{C}^N$ for some $N \in \mathbb{N}$

Proof. Suppose A is a commutative matrix algebra. Then $a_i b_i = b_i a_i$ for each $a_i, b_i \in M_{n_i}(\mathbb{C})$, with $i \in \{1, \ldots, N'\}$. This implies that each $a_i \in M_{n_i}(\mathbb{C})$ is diagonal and thus:

$$M_{n_i}(\mathbb{C}) \simeq \mathbb{C} \underbrace{\oplus \cdots \oplus}_{n_i \text{ times}} \mathbb{C}$$

Hence,

$$A \simeq \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{\Sigma_i n_i \text{ times}}$$

Defining $\sum_i n_i = N$ we arrive at $A \simeq \mathbb{C}^N$. □

2.3 Representation Theory

Definition 2.4: Invariant subspace

Let $L \in L(\mathcal{H})$, then we say that $V \subset \mathcal{H}$ is an *invariant subspace* of $\mathcal{H}$, if for all $v \in V$, $Lv \in V$.

Definition 2.5: Representation

Let $\mathcal{A}$ be a finite algebra. A *$*$-algebra representation* is a pair $(\mathcal{H}, \pi)$ where π is a $*$-algebra homomorphism $\pi : \mathcal{A} \to L(\mathcal{H})$. We say that the representation $(\mathcal{H}, \pi)$ is *irreducible* if the only invariant subspaces of $\pi(A)$ are $\mathcal{H}$ and $\{0\}$. If not, we say the representation is *reducible*

Example 6. *Example of representations*

Lemma 2.1: Shur's Lemma

Given a representation (H,π) of a $*$-algebra $\mathcal{A}$, with the *commutant* $\pi(\mathcal{A})'$ of $\pi(\mathcal{A})$ defined as:

$$\pi(\mathcal{A})' = \{T \in L(H) : \pi(a)T = T\pi(a) \text{ for all } a \in \mathcal{A}\}$$

Then a representation (H,π) of $\mathcal{A}$ is irreducible if and only if the commutant $\pi(\mathcal{A})'$ of $\pi(\mathcal{A})$ consists of multiples of the identity.

Proof. ($\Rightarrow$) Let $(\mathcal{H}, \pi)$ be an irreducible representation of $\mathcal{A}$. Then the only invariant subspaces of $\mathcal{H}$ under $\pi(a)$ are $\{0\}$ and $\mathcal{H}$ for all $a \in \mathcal{A}$. Take and element $T \in \pi(\mathcal{A})'$, then the range of T is:

$$R(T) = \{v \in \mathcal{H}; v = Tw \text{ for some } w \in \mathcal{H}\}$$

and the null space is given by:

$$N(T) = \{w \in \mathcal{H}; Tw = 0\}$$

Now, let us look at the images of these sets under our irreducible representation of $\mathcal{A}$. We see that $\pi(a)(R(T)) = \pi(a)T(\mathcal{H}) = T\pi(a)(\mathcal{H}) \in R(T), \forall a \in \mathcal{A}$ and $\pi(a)Tw = T\pi(a)w = 0, \forall a \in \mathcal{A}, w \in N(T)$. This means that both $N(T)$ and $R(T)$ are invariant subspaces of $\mathcal{H}$ under $\pi(a)$. We now have two options:

(i) $R(T) = \{0\}$ and $N(T) = \mathcal{H}$

(ii) $R(T) = \mathcal{H}$ and $N(T) = \{0\}$

Looking at condition (i) we see this is satisfied by the trivial operator $T = 0$. Instead, let us look at condition (ii). In this case, as $\mathcal{H}$ is finite, the operator T is invertible. By the invertible matrix theorem, there exists a non-zero value $\lambda \in \mathbb{C}$ such that $Tv = \lambda v$. We then have that $\det(T - \lambda I_{\mathcal{H}}) = 0$ and therefore $T - \lambda I_{\mathcal{H}}$ is not-invertible $\Rightarrow R(T - \lambda I_{\mathcal{H}}) \neq \mathcal{H}$ and $N(T - \lambda I_{\mathcal{H}}) \neq \{0\}$. But, $[T - \lambda I_{\mathcal{H}}, \pi(a)] = 0, \forall a \in \mathcal{A}$. Therefore, by the above argument, we have another two options:

(i) $R(T-\lambda I_{\mathcal{H}}) = \{0\}$ and $N(T-\lambda I_{\mathcal{H}}) = \mathcal{H}$

(ii) $R(T-\lambda I_{\mathcal{H}}) = \mathcal{H}$ and $N(T-\lambda I_{\mathcal{H}}) = \{0\}$

But we have shown that condition (ii) is not possible for this operator. Thus, $T = \lambda I_{\mathcal{H}}$.

($\Leftarrow$) Let $\pi(\mathcal{A})' = \{\lambda I_{\mathcal{H}}; \lambda \in \mathbb{C}\}$. Now, let $V \subset \mathcal{H}$ be an invariant subspace with respect to $\pi(a)$. We then decompose $\mathcal{H}$ as $\mathcal{H} = V \oplus V^{\perp}$ where $V^{\perp} = \{v \in \mathcal{H}; (v,w) = 0 \text{ for all } w \in V\}$. From $(\pi(a)v^{\perp}, v) = (v^{\perp}, \pi(a^*)v) = 0, \forall v \in V, v^{\perp} \in V^{\perp}$, thus we have that $V^{\perp}$ is also an invariant subspace.

Define the projection map $\text{proj}_V : \mathcal{H} \to V$. Then, $(\pi(a) \circ \text{proj}_V)(v) = \pi(a)(v) = (\text{proj}_V \circ \pi(a))(v)$, $\forall\ v \in \mathcal{V}$ and $(\pi(a) \circ \text{proj}_V)(v^{\perp}) = 0 = (\text{proj}_V \circ \pi(a))(v^{\perp})$, $\forall v^{\perp} \in V^{\perp}$. Therefore, $\text{proj}_V \in \pi(\mathcal{A})'$ and $\text{proj}_V = \lambda I_{\mathcal{H}}$ for some $\lambda \in \mathbb{C}$. This means that $V = \mathcal{H}$ or $V = \{0\}$. Thus, $(\mathcal{H}, \pi)$ is irreducible.

□

3 Lecture 3

We start by finishing the proof of Shur's Lemma and then finish off the arrow:

$$\text{Finite Topological Spaces} \longleftarrow \text{Commutative Matrix Algebras}$$

After this, we hope to generalise the notion to non-commutative matrix algebras.

3.1 Equivalence Classes

Definition 3.1: Equivalence relation

Let X be a set with binary relation $\sim$. We say that $\sim$ is an equivalence relation if:

(i) $a \sim a$ for all $a \in X$.

(ii) $a \sim b$ if and only if $b \sim a$ for all $a, b \in X$.

(iii) If $a \sim b$ and $b \sim c$ then $a \sim c$ for all $a, b, c \in X$.

We call the set X with an equivalence relation $\sim$ a *setoid*.

Definition 3.2: Unitary

Let U be a linear operator on $\mathcal{H}$. We say that U is *unitary* if $UU^* = U^*U = I_{\mathcal{H}}$.

Proposition 3.1

Given a $*$-algebra $\mathcal{A}$, the relation, $\sim$, defined by $(H_1, \pi_1) \sim (H_2, \pi_2)$ if there exists a unitary matrix, $U : H_1 \to H_2$ such that $\pi_1(a) = U^*\pi_2(a)U$, $\forall a \in \mathcal{A}$, is an equivalence relation on the set of representations of $\mathcal{A}$.

Proof. *(Reflectivity)* Given a representation (H_1, π_1), the identity map $I_{H_1} : H_1 \to H_1$ is clearly unitary and $\pi_1(a) = I_{H_1}\pi_1(a)I_{H_1}$, $\forall a \in \mathcal{A}$. Thus, $(H_1, \pi_1) \sim (H_1, \pi_1)$.

(Symmetry) Suppose $(H_1, \pi_1) \sim (H_2, \pi_2)$, then there exists a unitary matrix, U, such that $\pi_1(a) = U^*\pi_2(a)U$, $\forall a \in \mathcal{A}$. Then, $U\pi_1(a)U^* = UU^*\pi_2(a)UU^* = I_{H_2}\pi_2(a)I_{H_2} = \pi_2(a)$, $\forall a \in \mathcal{A}$, where I_{H_2} is the identity map on H_2. Define $\tilde{U} = U^* \Rightarrow \tilde{U}^* = U$, we have that $\pi_2(a) = \tilde{U}^*\pi_1(a)\tilde{U}$, $\forall a \in \mathcal{A}$. Thus, $(H_2, \pi_2) \sim (H_1, \pi_1)$.

(Transitivity) Suppose $(H_1, \pi_1) \sim (H_2, \pi_2)$ and $(H_2, \pi_2) \sim (H_3, \pi_3)$. Then, there exists two unitary matrices, $U, \tilde{U}$, with $U : H_1 \to H_2$ and $\tilde{U} : H_2 \to H_3$, such that $\pi_1(a) = U^*\pi_2(a)U$ and $\pi_2(a) = \tilde{U}^*\pi_3(a)\tilde{U}$, $\forall a \in \mathcal{A}$. We now have, $\pi_1(a) = U^*\tilde{U}^*\pi_3(a)\tilde{U}U$. Now, it is the case that $(\tilde{U}U)^* = U^*\tilde{U}^*$ so we simply need to show that $\tilde{U}U$ is unitary. So, $(\tilde{U}U)(U^*\tilde{U}^*) = \tilde{U}(UU^*)\tilde{U}^* = \tilde{U}I_{H_2}\tilde{U}^* = \tilde{U}\tilde{U}^* = I_{H_3}$ and, $(U^*\tilde{U}^*)(\tilde{U}U) = U^*I_{H_2}U = U^*U = I_{H_1}$. Hence, $(H_1, \pi_1) \sim (H_3, \pi_3)$.

□

Remark 9. If $(H_1, \pi_1) \sim (H_2, \pi_2)$ we say that the representations are *unitarily equaivalent*.

Proposition 3.2

If (H_1, π_1) is an irreducible representation of a $*$-algebra $\mathcal{A}$, and $(H_1, \pi_1) \sim (H_2, \pi_2)$, then (H_2, π_2) is also an irreducible representation of $\mathcal{A}$.

Proof. Assume towards a contradiction that (H_2, π_2) is not an irreducible representation of $\mathcal{A}$. Then there exists a set $V' \subset H_2$ such that $\pi_2(a)(V') \subset V'$, $\forall a \in \mathcal{A}$. As $(H_1, \pi_1) \sim (H_2, \pi_2)$ there exists a unitary matrix, $U : H_1 \to H_2$, such that $\pi_1(a) = U^*\pi_2(a)U$, $\forall a \in \mathcal{A}$. Define a new set $V = U^*(V') \subset H_1$ which is neither $\mathcal{H}_1$ nor $\{0\}$, due to U being invertible. Then, $\pi_1(a)(V) = (U^*\pi_2(a))(V') \subset U^*(V') \subset V$, $\forall a \in \mathcal{A}$. Contradiction as (H_1, π_1) is irreducible. Thus, (H_2, π_2) must also be irreducible. □

Definition 3.3: Equivalence class

Let X be a setoid. The we call the set $[a] = \{x \in X; x \sim a\}$ the *equivalence class* of a.

3.2 Structure Space and Finite Commutative Geometry

Definition 3.4

he *structure space*, $\hat{\mathcal{A}}$, of a $*$-algebra $\mathcal{A}$, is the set of equivalence classes of irreducible representations of $\mathcal{A}$. i.e.

$$\hat{\mathcal{A}} = \{[(H_i, \pi_i)] : i \in I\}$$

where (H_i, π_i) is an irreducible representation of $\mathcal{A}$ for each $i \in I$ and I is some indexing set.

Proposition 3.3

Any irreducible representation of a commutative algebra $\mathcal{A}$ is 1-dimensional.

Proof. Let $(\mathcal{H}, \pi)$ be an irreducible representation of a commutative algebra $\mathcal{A}$. Since $\mathcal{A}$ is commutative, we have that $\pi(a)\pi(b) = \pi(b)\pi(a), \forall a, b \in \mathcal{A}$. Then, $\pi(A) \subset \pi'(A)$. Using $\pi(\lambda 1_{\mathcal{A}}) = \lambda I_H$, with $\lambda \in \mathbb{C}$ and $1_{\mathcal{A}} \in \mathcal{A}$, and Schur's Lemma, we have that $\pi'(A) \subset \pi(A) \Rightarrow \pi(A) = \{\lambda I_H; \lambda \in \mathbb{C}\}$. Thus, any irreducible representation of a commutative algebra is 1-dimensional.

□

Proposition 3.4

Any irreducible representation of a commutative matrix algebra, $\mathcal{A}$, is of the form:

$$\pi_i : \mathbb{C}^N \ni (\lambda_1, \ldots, \lambda_N) \mapsto \lambda_i \in \mathbb{C}$$

Proof. By proposition 3.8, we have that all irreducible representations of $\mathcal{A}$ are of the form $\pi : \mathcal{A} \to \mathbb{C}$. Take a general representation:

$$\tilde{\pi} : \mathbb{C}^N \to \mathbb{C}$$

$$\tilde{\pi}\Big(\sum_{i=1}^{N} \lambda_i e_i\Big) = \lambda_j e'_j$$

for some $j \in \{1, \ldots, N\}$ and $\{e_i\}$ a basis for $\mathbb{C}^N$. Using that $\tilde{\pi}$ is a homomorphism, we have that $e'_j = 1$. □

Proposition 3.5

Let A be a commutative matrix algebra and $(\mathcal{H}, \pi_i)$ be a collection of irreducible representations of $\mathcal{A}$ defined above. Then no two irreducible representations are unitarily equivalent.

Proof. Let π_1 and π_2 be representations of $\mathcal{A}$ as defined in proposition 3.9. Recall, if π_1 is to be unitarily equivalent to π_2 there must exist a $U : \mathcal{H} \to \mathcal{H}$ such that $\pi_1(a) = U^*\pi_2(a)U, \forall a \in A$. Suppose towards a contradiction such a operator exists. But, $\pi_2(a)$ commutes with any linear operators on $\mathcal{H}$. Thus, $\pi_1(a) = \pi_2(a)$ for all $a \in \mathcal{A}$. Contradiction. So we must have that no two irreducible representations of $\mathcal{A}$ are unitarily equivalent. □

Definition 3.5: Coarseness/fineness

Let τ and τ' be two topologies on a set X. If $\tau \subset \tau'$ we say that τ is *coarser* than τ or alternatively τ' is *finer* than τ.

Theorem 3.1

Let A be a commutative matrix algebra. Then the structure space $\hat{A}$, with weak $*$-topology, is isomorphic to a finite topological space X, with discrete topology, and $A \simeq C(X)$.

So, what was the point in all this? Our goal was to successfully relate finite commutative matrix algebras to finite topological spaces. Using propositions 2.7 and 2.8 it should be clear that the structure space of $\mathcal{A}$ has the same cardinality as a finite set $X = \{1, \ldots, N\}$ and thus, $\mathcal{A} \simeq X$. Notice that at this point we have not imposed any topological structure on the space X (in fact, there is no structure on $\hat{\mathcal{A}}$ and therefore X at all). So, the natural question to ask is can we introduce a topology onto $\hat{\mathcal{A}}$? The answer, of course, is yes. But, what topology should be impose on $\hat{\mathcal{A}}$? Well, in our initial construction of $\mathcal{A}$ we wished for $\mathcal{A} \simeq C(X)$, where $C(X)$ is the space of continuous functions on X. But, to ensure that $C(X)$ also forms an algebra, with point-wise structure, we have to impose the condition that every function on X to $\mathbb{C}$ is continuous. So, our questions reduces to, what is the coarsest topology we can impose on X such that every element of $\{f; f : X \to \mathbb{C}\}$ is continuous. This is called the weak topology and in the finite case reduces to the discrete topology on X as required. We will discuss the weak topology in more detail later.

3.3 Fields, Rings and Modules

Definition 3.6: Module

Let $\mathcal{A}$ be an algebra of a field $\mathbb{K}$. A *left $\mathcal{A}$-module* is a vector space E, over the field $\mathbb{K}$, with a bilinear product $\mathcal{A} \times E \ni (a, e) \mapsto a \cdot e \in E$ such that:

$$(i) \quad (a_1 a_2) \cdot e = a_1 \cdot (a_2 \cdot e) \quad \forall a_1, a_2 \in \mathcal{A}, \forall e \in E$$

$$(ii) \quad I_{\mathcal{A}} \cdot e = e, \quad \text{where } I_{\mathcal{A}} \text{ is the identity in } \mathcal{A} \text{ and } e \in E$$

4 Lecture 4

The aim of this lecture is to tie up the algebraic classification of the geometry of a finite number of points. We have already seen how to translate to and from algebra and geometry when we are talking about finite topological spaces. Now we want to look at something a bit more useful in physical situations. That is the notion of a distance, and specifically how can we describe algebraically the distance between points in our finite topological spaces.

4.1 Finite Metric Spaces

Definition 4.1: Metric

Let X be a set. A metric on X is a map $d : X \times X \to \mathbb{R}$ such that:

(i) $d(x, y) = 0$ iff $x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

We refer to the pair (X, d) as a metric space. Like we did with topological spaces, we will often denote the pair as simply X.

We think of the map $d(x, y)$ as the distance between the points x and y in the set. It can be shown by the above axioms that the metric is always positive.

Definition 4.2: Balls

Let X be a metric space. We define the set $B_\epsilon(c) = \{x \in X;\ d(c,x) < \epsilon, c \in X, \epsilon > 0\}$ as the open ball with centre c and radius ϵ.

Proposition 4.1

Let X be a metric space. Then the set:

$$\tau_d = \{U \subseteq X; \text{for all } x \in U, \text{ there exists } \epsilon > 0 \text{ such that } B_\epsilon(x) \subseteq U\}.$$

is a topology on X. We call the topology inherited from a metric the metric topology.

Proof. Will fill this out later □

It is worth noting that not all topologies are metric topologies. In other words, not all topologies can be realised from distance functions on the space. This is exactly the reason why one first introduces the notion of open sets without introducing the distance function as it is done in real analysis.

For a finite discrete space, a metric is described by a collection of real non-negative numbers, $\{d_{ij}\}_{i,j \in X}$. The following proposition tells us exactly how to algebraically encode this data in the setting of noncommutative geometry (despite still being commutative).

4.2 Finite Dirac Operator

A useful notion for the following statement is that of a norm and specifically that of an operator norm, which are used fairly commonly throughout modern mathematics. General definition of a norm

Definition 4.3: Operator # 2

Let $A: V \to W$ be an operator between two normed vector spaces then the operator norm is defined as $\|A\|^2 = \sup_{v \in H}\{(Av, Av) \quad s.t \quad (v,v) \leq 1\}$.

Theorem 4.1

Let d_{ij} be a metric on the space X of N points and set $A = \mathbb{C}^N$ with elements $a = (a(i))_{i=1}^N$, so that $\hat{A} = X$. Then there exists a representation $\pi: A \to L(H)$ on a finite dimensional inner product space H and there exists a symmetric operator $D: H \to H$ such that

$$d_{ij} = \sup_{a \in A}\{|a(i) - a(j)| \ : \ \|[D, \pi(a)]\| \leq 1\}$$

Proof. The tricky part is to show that

$$\|[D, \pi(a)]\| = \max_{k \neq l}\{\frac{1}{d_{kl}}|a(k) - a(l)|\} \tag{1}$$

and once this is done the proof is more straight forward. So assuming it to be true then the condition $\|[D, \pi(a)]\| \leq 1$ is just $\max_{k\neq l}\{\frac{1}{d_{kl}}|a(k) - a(l)|\} \leq 1$. So we have that $\frac{1}{d_{kl}}|a(k) - a(l)| \leq 1$

for any k, l. Specifically $|a(k) - a(l)| \leq d_{kl}$, and taking the supremum of this we get the condition that

$$\sup_{a \in A}\{|a(i) - a(j)| : \|[D, \pi]\| \leq 1\} \leq d_{ij}$$

We then have to show that $\sup_{a\in A}\{|a(i) - a(j)| : \|[D, \pi]\| \leq 1\} \geq d_{ij}$ and thus we have equality. To show this we fix i, j and take $a \in A$ to be such that $a(k) = d_{ik}$. Then we have that

$$|a(i) - a(j)| = |\underbrace{d_{ii}}_{=0} - d_{ij}| = d_{ij}$$

so that $\frac{1}{d_{kl}}|a(k) - a(l)| = \frac{1}{d_{kl}}|d_{ik} - d_{il}| \leq \frac{|d_{kl}|}{d_{kl}} \leq 1$, where the inequality arises by property 4 of a metric. So now all that is left to show is that Eq (1) is true, which we will now do using induction.

Let $N = 2$ so that $H = \mathbb{C}^2$ then let π and D take the following forms:

$$\pi(a) = \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix}, \qquad D = \frac{1}{d_{12}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So we then have that $\|[D, \pi(a)]\| = (d_{12})^{-1}|a(1) - a(2)|$, where the norm is defined as $\|A\|^2 = \sup_{v\in H}\{(Av, Av) : (v, v) \leq 1\}$. So now we have proved the equality holds for $N = 2$ assume it holds for an arbitrary N and we want to show it also holds for $N + 1$. So assume we have representation π_N of $\mathbb{C}^N$ onto an inner product space H_N and a symmetric operator D_N. And define

$$H_{N+1} = H_N \oplus \bigoplus_{i=1}^{N} H_N^i$$

where $H_N^i = \mathbb{C}^2$. We then use the $N = 2$ case as motivation for our representation:

$$\pi_{N+1}(a(1), a(2), \ldots, a(N+1)) := \pi_N(a(1), \ldots a(N)) \oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(N+1) \end{pmatrix} \oplus \begin{pmatrix} a(2) & 0 \\ 0 & a(N+1) \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} a(N) & \\ 0 & a(\end{pmatrix}$$

and let D be the following:

$$D_{N+1} := D_N \oplus \frac{1}{d_{1(N+1)}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \frac{1}{d_{N(N+1)}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus we need to look at $[D_{N+1}, \pi_{N+1}(a)]$ and its norm:

$$[D_{N+1}, \pi_{N+1}(a)] = [D_N, \pi_N(\tilde{a})] \tag{2}$$

$$\oplus \frac{1}{d_{1(N+1)}}\begin{pmatrix} 0 & a(N+1) - a(l) \\ a(1) - a(N+1) & 0 \end{pmatrix} \oplus \ldots \tag{3}$$

$$\oplus \frac{1}{d_{N(N+1)}}\begin{pmatrix} 0 & a(N+1) - a(N) \\ a(N) - a(N+1) & 0 \end{pmatrix} \tag{4}$$

If we take a vector v in H then we have:

$$([D_{N+1}, \pi_{N+1}(a)]v, [D_{N+1}, \pi_{N+1}(a)]v) = \sum_{i<j\in\{1,2,\ldots N+1\}} \frac{1}{d_{ij}^2}|a(i) - a(j)|^2(v_{ij}^2 + v_{ji}^2) \tag{5}$$

where v_{ij} corresponds to the terms in (4) between points i, j. Now we need to show that the supremum of this inner product with respects to vectors v such that $(v, v) \leq 1$ is precisely given by the right hand side of (1). To do we factor out of the sum the maximum so that:

$$\sum_{i \neq j \in \{1,2,\ldots N+1\}} \frac{1}{d_{ij}^2} |a(i) - a(j)|^2 v_{ij}^2 = \max_{k \neq l} \{ \frac{1}{d_{kl}^2} |a(k) - a(l)|^2 \} \sum_{i<j \in \{1,2,\ldots,N+1\}} \alpha_{ij} v_{ij}^2 \tag{6}$$

where the coefficient $\alpha_{ij} \leq 1$ for all i, j. Now utilising the definition of the operator norm we have that:

$$\begin{aligned} \|[D_{N+1}, \pi_{N+1}(a)]\|^2 &= \sup_{v \in H} \{ \max_{k \neq l} \{ \frac{1}{d_{kl}^2} |a(k) - a(l)|^2 \} \sum_{i<j \in \{1,2,\ldots,N+1\}} \alpha_{ij} v_{ij}^2 \} && (7) \\ &= \max_{k \neq l} \{ \frac{1}{d_{kl}^2} |a(k) - a(l)|^2 \} \sup_{v \in H} \{ \sum_{i<j} \alpha_{ij} v_{ij}^2 \} && (8) \end{aligned}$$

As $\alpha_{ij} \leq 1$ for all i, j, we have that the sum is less than or equal to $(v, v) \leq 1$ and therefore the supremum is that it is equal to one. So we have the following:

$$\|[D_{N+1}, \pi_{N+1}(a)]\| = \max_{k \neq l} \{ \frac{1}{d_{kl}^2} |a(k) - a(l)|^2 \}$$

□

5 Lecture 5

5.1 Non-finite Stuff

Aim to introduce people to the ideas of spin geometry and aim to describe the real spectral triple for a Riemannian spin manifold. Where to begin? Begin at the beginning of course!

A fundamental object we require is that of a manifold. Many of the structures we can layer on top of a manifold are highly useful in physics, and are basically applicable in every area. Such as string theory, quantum field theory, condensed matter physics, cosmology. The list is endless so its worth recapping what we need. We are going to work with smooth manifolds, however many results hold for less regular manifolds. But we're lazy, and don't want to think about whether we have sufficient regularity. I only know how to define a top manifold, then a differential structure etc. Is there a straight way to get to smooth? I guess not but thought i'd ask

Definition 5.1: Manifold

A topological manifold is a topological space M, that is:

1. *Hausdorff:* $\forall x, y \in M$ we can find open sets U, V, such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
2. *Second countable:* There exists a countable basis of open sets in the topology.
3. *Locally Euclidean:* For every point $x \in M$ there exists a neighbourhood, U such that there is a homeomorphism, $\phi : U \to V \subset \mathbb{R}^n$, for some fixed natural number n.

In order to do calculus we need to add on some extra structure. Namely a smooth structure.

Definition 5.2: Smoothness

A smooth structure on a topological manifold is a collection of pairs $\{(U_a, \phi_a)\}$, which satisfy the following

1. The U_a are open sets that cover the manifold. ($\bigcup_a U_a = M$)
2. For each a, we have that $\phi_a : U_a \to \mathbb{R}^n$ is a homeomorphism.
3. The transitions functions $t_{ab} := \phi_b \circ \phi_a^{-1} \colon \phi_a(U \cap V) \to \phi_b(U \cap V)$ are smooth in the usual sense as a map from $\mathbb{R}^n$ to itself.

We call ϕ_a charts and the collection $\{(U_a, \phi_a)\}$ an atlas.

Could do with a diagram, but will get around to that later A manifold with a smooth structure is called a smooth manifold. Let $C^\infty(M) = \{f \colon M \to \mathbb{R} | f \circ \phi^{-1} \text{ is smooth } \forall \text{ charts } \phi\}$ be known as the algebra of smooth functions over a manifold. Where the operations needed to make it an algebra are defined pointwise. We will now describe the Tangent bundle on a manifold and then what a fibre bundle is.

Definition 5.3: Derivations

derivation at the point $x \in M$ is a linear map $D : C^\infty(M) \to \mathbb{R}$ that has the following property at each point on the manifold:

$$D(fg) = D(f) \cdot g(x) + f(x) \cdot D(g)$$

Definition 5.4: Tangent Space

Given a point on a manifold M we can define the tangent space at that point as the real vector space of derivations, where we have the following operations: $(D_1 + D_2)(f) = D_1(f) + D_2(f)$ and $(\lambda D)(f) = \lambda D(f)$ for $\lambda \in \mathbb{R}$. We will denote the tangent space in the usual way as $T_x(M)$.

The notion of a tangent bundle is now a way to stitch together the tangent spaces at each point. We will define a more general bundle structure and show where tangent bundles fit into that framework. However it should be said that a tangent bundle always exists for a smooth manifold, and does not rely on any extra structure, such as metrics.

Definition 5.5: Tangle Bundle

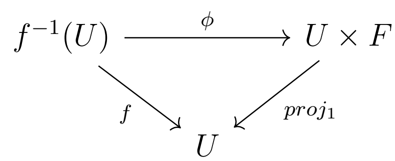
Given a smooth manifold M, the *tangent bundle* denoted TM, is defined as a set $TM = \bigsqcup_{x \in M} T_x M = \bigcup_{x \in M} \{x\} \times T_x M$ combined with the map $\pi \colon TM \to M$ defined as a projection onto the first component.

You can think of elements of the tangent bundle as being (x, v) where x is a point on the manifold and v is an element of $T_x M$. The set TM inherits a topology from M, and thus is a topological space and it turns out to also be a manifold. But I'm not going to show this, there is a lot of literature on the tangent bundle and other bundles that work through all the details. However, it is worth mentioning that the tangent bundle is an example of a vector bundle over a manifold. And to describe bundles of manifolds the most useful definition is that of a fibre bundle which is included for use later.

Definition 5.6: Fibre bundle

A *fibre bundle* is a structure (E, M, F, f), where E, M, F are all topological spaces and $f\colon E \to M$ is a continuous surjection that satisfies the following properties: $\forall x \in E$ there exists a neighbourhood U of $f(x) \in M$ such that there exists a homeomorphism $\phi : f^{-1}(U) \to U \times F$ which, when composed with projection onto the first component, agrees with f.

The condition for f can be summarised in the requiring that the following diagram commutes:



To proceed in our description of spaces by algebras we are going to need some differential geometry that is probably unfamiliar. Namely the notion of a *spin bundle*. Once we have all the notions of a spin bundle we can then construct a real spectral triple for a Riemannian spin manifold.

Need metric tensor and notion of Riemannian

6 Lecture 6

In my half of the lectures, we will be covering the topic of C^*-algebras. This can be encased into the larger and broader topic of operator algebras. Given that this part of the seminar may not be as intuitive as the geometrical part, I will give a historical overview of why one should be interested in operator algebras from a physics perspective.

6.1 History/Motivation

We start by asking the question, Why should one care about operator algebras? The obvious answer for us is because we wish to generalise ordinary differential geometry to something non-commutative in order to geometrically interpret quantum mechanical systems. Perhaps the most natural case of operator algebras appearing is in the canonical quantisation used in quantum mechanics. Here is a quick overview of the construction used.

Classical Mechanics: We start by being given all the initial configurations (positions) of our particles. This is written mathematically as picking an open set $U \in \mathbb{R}^N$ with elements of U being the positions of the particles. One then constructs the contangent bundle of this space, known as the phase space, which has the property of being symplectic. With this property, one is then able to construct a Poisson bracket on the phase space through the symplectic form. Via a Hamiltonian function, one can then describe the dynamics of the system as a series of differential equations given by the Hamiltonian flow (or just Hamiltons equations).

Quantum Mechanics (Dirac): With the above construction one then constructs a quantisations linear map $Q : C(M) \to \mathrm{End}(\mathcal{H})$ satisfying the following properties:

- $[Q_f, Q_g] = i\hbar Q_{\{f,g\}}$

- $Q_{g \circ f} = Q_g \circ Q_f$
- $Q_x \psi = x\psi$ and $Q_p \psi = -i\hbar \partial_x \psi$

In other words, one replaces to algebra of continuous functions on the phase space, with a non-commutative algebra of bounded operators on a Hilbert space.

Theorem 6.1: Gelfand-Naimark Theorem

Let $\mathcal{A}$ be a commutative C^*-algebra. Then the Gelfand transform:

$$\Gamma : A \to C_0(\Omega(A))$$

is an isometric $*$-isomorphism.

Theorem 6.2: Gelfand-Naimark-Segal Theorem

Let A be a C^*-algebra. Then its universal representation is faithful.

These are the two main results we hope to prove throughout the next few weeks and are generalisations of the earlier theorems we proved in the finite case. In short, the first theorem allows us to realise that the space of characters of an algebra (or maximal ideals, or pure states) can be thought of as a (locally) compact, Hausdorff topological space. The second tells us that an arbitrary C^*-algebra can be realised as a sub-algebra of bounded operators on a Hilbert space. Through this, one can then realise that the algebra A is a von-Neumann algebra. To the well versed category theorist, this is must succinctly written as: There exists an invertible, contravariant functor between the opposite category of C^*-algebras and the category of locally compact, Hausdorff topological spaces.

6.2 C^*-Algebras

Definition 6.1: Cauchy sequence

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). We say that $(x_n)_{n\in\mathbb{N}}$ is a *Cauchy sequence* if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > N$.

Definition 6.2: Completeness

We say that a metric space (X, d) is *complete* if every Cauchy sequence in (X, d) converges to a point in X.

Example 7. *An obvious example of a complete space, is the space $\mathbb{R}^n$ with the usual metric defined on it.*

Example 8. *Perhaps more interesting is an example of a non-complete space. The open interval $(0, 1)$ is not complete with respect to the metric inherited from $\mathbb{R}$, same for $\mathbb{Q}$.*

Definition 6.3: Inner product

Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. An *inner product* on $\mathcal{V}$ is a map $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ such that:

(i) $\langle v, u \rangle = \overline{\langle u, v \rangle}$ for $v, u \in \mathcal{V}$

(ii) $\langle \alpha v, u \rangle = \alpha \langle v, u \rangle$ and $\langle v + u, w \rangle = \langle v, w \rangle + \langle u, w \rangle$ for $\alpha \in \mathbb{F}, u, v, w \in \mathcal{V}$

(iii) $\langle v, v \rangle \geq 0,\ v \in \mathcal{V}$ and $\langle v, v \rangle = 0$ iff $v = 0$

We refer to the pair $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ as a *inner product space*. An inner product defines a metric via the following relation $d(v, u) = \sqrt{\langle v - u, v - u \rangle}$.

Definition 6.4: Hilbert space

We refer to a complete inner product space as a *Hilbert space*.

Example 9. *The example most used is the space of square integrable functions. This can be formalised as the space of Lesbesgue integrable functions. Alternatively, we can say it is a real (complex) valued measurable function such that:*

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty$$

With inner product:

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) dx$$

The space $L^2(\mathbb{R})$ is a Hilbert space under this inner product.

Definition 6.5: Norm

A *norm* on a vector space $\mathcal{V}$ is a map $|| \cdot || : \mathcal{V} \to [0, \infty)$ such that:

(i) $||v|| = 0$ iff $v = 0$

(ii) $||\lambda v|| = |\lambda| ||v||$ for $\lambda \in \mathbb{F}, v \in \mathcal{V}$

(iii) $||v + u|| \leq ||v|| + ||u||,$ for $v, u \in \mathcal{V}$

We refer to the pair $(\mathcal{V}, || \cdot ||)$ as a *normed space*. A norm on $\mathcal{V}$ defines a metric via the following relation, $d(v, u) = ||v - u||$. A norm that only satisfies conditions (i) and (ii) is refered to as a *semi-norm*.

Definition 6.6: Banach space

We refer to a complete normed space as a *Banach space*.

Remark 10. From the above definitions it is clear that following statements are true. All Hilbert Spaces are Banach Spaces and all Banach Spaces/Hilbert Spaces are metric spaces.

Definition 6.7

A *normed algebra* is an algebra with an inner product defined on $\mathcal{V}$ such that:

$$||ab|| \leq ||a||||b||, \quad \forall a, b \in \mathcal{A}$$

Definition 6.8: Banach algebra

A complete normed algebra is referred to as a *Banach algebra*.

Definition 6.9: C^*-algebra

A *C^*-algebra* is a Banach algebra with an involution such that:

$$||a^*a|| = ||a||^2, \quad \forall a \in \mathcal{A}$$

Proposition 6.1

Let E be a normed vector space and F be a Banach space. Then the space of bounded operators from E to F, denoted $B(E, F)$ is a Banach space.

Remark 11. By the above proposition, this tells us that the dual space of a vector space it always a Banach space.

Definition 6.10: Operator # 3

Let $(\mathcal{U}, ||\cdot||_1)$ and $(\mathcal{V}, ||\cdot||_2)$ be normed vector spaces over a field $\mathbb{F}$. Then we say that the operator $L : \mathcal{U} \to \mathcal{V}$ is *bounded* if there exists $M > 0$ such that $||Lu||_2 \leq M||u||_1$ for all $u \in \mathcal{U}$.

Remark 12. We denote the space of bounded linear operators from normed space $\mathcal{U}$ to normed space $\mathcal{V}$ by $\mathcal{B}(\mathcal{U}, \mathcal{V})$.

Proposition 6.2

Let $(\mathcal{U}, ||\cdot||_1)$ and $(\mathcal{V}, ||\cdot||_2)$ be normed vector spaces over a field $\mathbb{F}$. For $L \in \mathcal{B}(\mathcal{U}, \mathcal{V})$, we claim that $||L||_{op} = \inf\{M > 0; ||Lu||_2 \leq M||u||_1\}$ is a norm on the space $\mathcal{B}(\mathcal{U}, \mathcal{V})$. We will refer to the norm defined above as the operator norm.

Remark 13. With the above result it is clear that $\mathcal{B}(\mathcal{U}, \mathcal{V})$ is a normed vector space. We gave the norm as the smallest M that satisfies the bounded condition. Loosely speaking, the the norm we defined is the length of the operator as it measures (to best accuracy) the change in length of the vectors. Maybe re-write this remark later!

Proposition 6.3

The following norms on $\mathcal{B}(\mathcal{U},\mathcal{V})$ are equivalent:

(i) $\inf\{M > 0; ||Lu||_2 \leq M||u||_1\}$

(ii) $\sup\{||Lu||; ||u|| \leq 1\}$

(iii) $\sup\{||Lu||; ||u|| = 1\}$

(iv) $\sup\{\frac{||Lu||}{||u||}; ||u|| \neq 0\}$

with $L \in \mathcal{B}(\mathcal{U},\mathcal{V})$ and $u \in \mathcal{U}$.

Remark 14. By the above proposition, all these norms are equivalent to the operator norm. We will hence label any of these norms as the operator norm, and use the one that allows for the simplest computations.

Theorem 6.3

Every finite dimensional normed space is a Banach space.

7 Lecture 7: Fibre bundles

Recall the definition of a fibre bundle (E, M, F, f). To define a smooth fibre bundle we require the topological spaces E, M, F to be smooth manifolds and f a smooth surjection. With the local trivialisation now being diffeomorphism. We will now look at different types of bundles that commonly appear in physics.

7.1 Bundles with symmetry

Definition 7.1: Principal G-bundle

Fibre bundle with structure groups are a smooth fibre bundle (E, M, F, f) with local trivialisations $\{(U_i, \phi_i)\}$ and let G be a Lie group with a left action on F denoted by ρ. For any intersecting pair of open sets U_i and U_j there is a smooth map $g_{ij} \colon U_i \cap U_j \to G$ such that we have that

$$\phi_j \circ \phi_i^{-1} \colon (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F \tag{9}$$

$$(x, \lambda) \to (x, \rho(g_{ij}(x))\lambda) \tag{10}$$

for all $(x, \lambda) \in (U_i \cap U_j) \times F$. We call the functions $\{g_{ij}\}$ the transition functions.

Now as Lie groups describe continuous symmetries we have a way to introduce the notion of a symmetry of the fibres F. A simple and common example of a fibre bundle with a structure group is where the fibres are the group itself, $F = G$. These are given a special name because of there usefulness and they are called *Principal G-bundles.* Another type of bundle with structure group G are the *associated G-bundles.* These are defined as follows:

Definition 7.2: Associated bundles

Given a representation of a Lie group G on a vector space V, denoted by $\rho : G \to End(V)$. Let us take $E' = (E \times V)/G$, which we will denote by $E' = E \times_G V$ and then (E', M, V, f, G) is called the associative bundle of (E, M, F, f, G).
We can actually set up a correspondence between vector bundles and principal g-bundles. Which we will describe below:

7.2 Constructing fibre bundles from local data

Let M be a smooth manifold and let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M. Let F be a smooth manifold with a left action, ρ, by a Lie group G. Given a set of smooth transition functions $g_{ij} : (U_i \cap U_j) \to G$ such that on any non-empty intersection $U_{ijk=}U_i \cap U_j \cap U_k$ we have that $g_ik(x) = g_ij(x)g_jk(x)$ for all $x \in U_{ijk}$. Then we can construct a fibre bunle (E, M, F, f) with structure group G by taking

$$E = (\bigsqcup_\alpha U_\alpha \times F)/\sim$$

where $(x, \lambda) \sim (x, \rho(g_{ij}(x))\lambda)$ and by taking π to be projection onto the first entry, $pr_1 \colon U_i \times F \to U_i$.

7.3 Frame bundle

Given a vector bundles (E, M, V, f) we can define a principal g-bundle $(GL(E), M, V, \pi)$ by taking $Gl(E)_x$ to be the set of frames[1] of V. As any two frames are related by a unique invertible linear transformation, which means that we can view $GL(E)_x = GL(E_x)$. Then $Gl(E_x)$ are the fibre of the fibre bundle $(GL(E), M, GL(E)_x, \pi)$ *constructed from local data* by taking $GL(E) = (\bigsqcup_\alpha U_\alpha \times GL(V))/\sim$ as described in the previous section.

7.4 When are two bundles the same?

We can compare bundles and specifically say when they are equivalent to each by using a *bundle morphism*.

Definition 7.3: Bundle morphism

A bundle morphism between two fibre bundles (E, M, F, f) and $(\tilde{E}, M\tilde{F}, \tilde{f})$ such that we have a smooth map $\psi \colon E \to \tilde{E}$ such that $\tilde{f} \circ \psi = f$ (or the following diagram commutes).
need commutative diagram.

[1] A frame at the point $x \in M$ is an ordered basis for the fibre E_x, which is a vector space

8 Lecture 8

8.1 The Gelfand-Naimark Theorem

We start where we left off, with the algebraic aspects of NCG. Recall our efforts are to prove the Gelfand -Naimark theorem, the corner stone of NCG.

Theorem 8.1: Gelfand-Naimark Theorem

Let $\mathcal{A}$ be a commutative C^*-algebra. Then the Gelfand transform:

$$\Gamma : A \to C_0(\Omega(A))$$

is an isometric $*$-isomorphism.

Example 10. *Let S be a set. The set $l^\infty(S)$ is the set of all bounded complex functions on S. This is a unital Banach algebra with norm $||f||_\infty = \sup_{x \in S} |f(x)|$.*

Example 11. *Let X be a locally compact Hausdorff space. The set of complex valued functions that vanish at infinity denoted $C_0(X)$ form a Banach algebra (as they are a subset of the above example). In fact, with the involution defined at complex conjugation, it is a C^*-algebra. The algebra is unital iff the topological space is compact. This is the guiding example for Gelfand-Naimark.*

Definition 8.1: Spectrum

Let A be an algebra. Given an element $a \in A$, we say that the *spectrum* of a is given by:

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1_A \notin \text{Inv}(A)\}$$

where $\text{Inv}(A)$ are the invertible elements of A.

Example 12. *The spectrum of the algebra $M_n\mathbb{C})$ is the eigenvalues of the matrix.*

Example 13. *Let X be a compact Hausdorff space and $C(X)$ the space of continious functions on X. The spectrum of $f \in C(X)$ is Spec(f) = Im(f) $= \{\lambda \in \mathbb{C} : f(x) = \lambda\}$*

Definition 8.2: Spectral radius

Let A be an algebra. We define the *spectral radius* of an element $a \in A$ as:

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$$

Theorem 8.2: Gelfand

If a is an element of a unital Banach algebra A, then the spectrum of a is non-empty.

Theorem 8.3: Gelfand-Mazur

If A is a unital Banach algebra in which every non-zero element is invertible, then $A = 1_A\mathbb{C}$.

Proof. Follows from Gelfand's theorem. □

Definition 8.3: Character

A character ψ on an algebra $\mathcal{A}$ is a non-trivial algebra homomorphism between $\mathcal{A}$ to $\mathbb{C}$. We denote the set of all characters by $\Omega(\mathcal{A})$.

Definition 8.4

Let V be a vector space. We say the space of linear maps from V to $\mathbb{C}$ is the *dual space* of V. The space is denoted by V^*.

Definition 8.5: Ideals

Let A be an algebra. We say that $I \subset A$, as a vector space, is a *right (left)* ideal if $ab \in I$ for $a \in A$ and $b \in I$ ($ba \in I, b \in I, a \in A$). We call a left-right ideal simply an *ideal.*

Definition 8.6

We say that an ideal is *proper*, if I is not A.

Definition 8.7: Maximal ideal

An ideal is *maximal* if it is not contained in any other proper ideal.

Lemma 8.1

If I is a modular maximal ideal of a unital commutative algebra A, then A/I is a field.

Theorem 8.4

Let A be a unital commutative Banach algebra.

(i) $\tau(a) \in \sigma(a)$ for every $\tau \in \Omega(A)$ and every $a \in A$

(ii) $||\tau|| = 1$ for all $\tau \in \Omega(A)$.

(iii) The set $\Omega(A)$ is non-empty, and the map, $\tau \mapsto \text{Ker}(\tau)$, defines a bijection from $\Omega(A)$ onto the set of maximal ideals of A.

Proof. (i) Follows from later theorem.

(ii) By theorem 8.18, $\tau(a) \in \sigma(a)$ for each $a \in A$. Thus, $|\tau(a)| < r(a) \leq ||a||$. Therefore, $||\tau|| \leq 1$ and by the homomorphism property, $\tau(1) = 1$. Thus, $||\tau|| = 1$.

(iii) Let I = Ker(τ) for some $\tau \in \Omega(A)$, $\tau \neq 0$ (exists as algebra is unital). This is an proper ideal of A as $\tau(ab) = \tau(a)\tau(b) = 0$ where $a \in A, b \in I$, likewise for the right action. Now, note that $a - \tau(a)1_A \in I$ for all $a \in A$. Thus, our algebra can be decomposed as $A = 1_A\mathbb{C} + I$, and hence I is maximal. We must now show the above map is bijective.
(Injective) Let $\tau_1, \tau_2 \in \Omega(A)$ and $\text{Ker}(\tau_1) = \text{Ker}(\tau_2)$. Note now that $\tau_1(a - \tau_2(a)1_A) = 0$ for each $a \in A$. Thus, $\tau_1 = \tau_2$.
(Surjective) We wish to show for any maximal ideal $I \subset A$, there exists $\tau \in \Omega(A)$ such that $I = \text{Ker}(\tau)$. So, suppose I is a maximal ideal of A(the existence of these ideals follows from Zorn's Lemma and the fact that A is unital). Then, as A is unital, I is modular, and by lemma 8.19, A/I is a field. Thus, by Gelfand-Mazur theorem, we can write the quotient algebra as $A/I = \mathbb{C}(1_A + I)$. This then implies that $A = \mathbb{C}1_A + I$. We now define the map $\tau : A \to \mathbb{C}$ such that $\tau(a + \lambda 1_A) = \lambda$. Therefore, we have shown the existence of a $\tau \in \Omega(A)$ such that I =Ker(τ). □

Theorem 8.5

Let A be a commutative Banach algebra.

(i) If A is unital, then $\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\}$

(ii) If A is non-unital, then $\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\} \cup \{0\}$

Proof. (i) Let $\lambda \in \sigma(a)$. Then the ideal $I = (a - \lambda)A$ is proper, as $1_A \notin I$. By theorem 8.20 (iii), I is contained in a maximal ideal Ker(τ), with $\tau \in \Omega(A)$. This means that $\tau(a) = \lambda$ for some $a \in A$, and thus $\sigma(a) \subseteq \{\tau(a) : \tau \in \Omega(A)\}$. It is clear that $\{\tau(a) : \tau \in \Omega(A)\} \subseteq \sigma(a)$, this follows from $\tau(\tau(a) - a) = 0$, and therefore equality holds.

(ii) Exercise □

Definition 8.8: Open ball topology

Let V be a vector space and $\{\rho_i\}_{i\in I}$ a family of semi-norms on V. Then the *open ball* at each i is given by:

$$B_r^i(u) = \{v \in V : \rho_i(u-v) < r, u \in V, r > 0\}$$

We refer to the topology generated by the set of open balls as the *topology induced by* $\{\rho_i\}_{i\in I}$

Definition 8.9: Weak $*$-topology

(Weak $*$-topology) Let V be a normed vector space. For each $v \in V$ we define a family of semi-norms on V^* by:

$$\rho_v(v^*) = |\langle v, v^* \rangle|$$

$v^* \in V^*$. The topology induced from this family of semi-norms if referred to as the *weak $*$-topology*.

Proposition 8.1

The weak $*$-topology is Hausdorff.

Theorem 8.6: Banach–Alaoglu theorem

Let V be a normed vector space. The closed unit ball defined by:

$$B^* = \{\rho \in V^* : ||\rho|| \leq 1\} \subseteq V^*$$

Then B^* is compact with respect to the weak $*$-topology.

Definition 8.10: Character space

From now on we endow take $\Omega(A)$ to be endowed with the weak $*$-topology, and refer to the total structure as the *character space*.

Theorem 8.7

If A is a commutative Banach algebra, then $\Omega(A)$ is a locally compact Hausdorff space. If A is unital, then $\Omega(A)$ is compact.

Proof. Let us start with the non-unital case. By theorem 8.16 it follows that the set $\Omega(A) \cup \{0\}$ is contained in the closed unit ball of A^*. By Banach–Alaoglu theorem it follows that the set is compact. Thus, $\Omega(A)$ is locally compact (see Alexandroff extension). The unital case follows by a similar argument using theorem 8.16. □

Definition 8.11: Gelfand Transform

Let A be a commutative Banach algebra for which $\Omega(A) \neq \emptyset$. We define the *Gelfand transform* as the map $\hat{a} : \Omega(A) \to \mathbb{C}$, $\tau \mapsto \tau(a)$. Note that $\hat{a} \in C_0(\Omega(A))$. This follows from the weak $*$-topology we endowed on $\Omega(A)$.

Theorem 8.8: Beurling

If a is an element of a unital Banach algebra A, then:

$$r(a) = \inf_{n\geq 1} ||a^n||^{1/n} = \lim_{n\to\infty} ||a^n||^{1/n}$$

Theorem 8.9: Gelfand Representation

Let A be a commutative Banach algebra and that $\Omega(A) \notin \emptyset$. Then the map:

$$A \to C_0(\Omega(A)), \quad a \mapsto \hat{a}$$

is a norm-decreasing homomorphism.

Proof. If A is unital, by theorem 8.21 it follows that $\sigma(a) = \hat{a}(\Omega(A))$ and if non-unital, $\sigma(a) = \hat{a}(\Omega(A)) \cup \{0\}$ for each $a \in A$. Therefore, $r(a) = ||\hat{a}||_\infty$ which tells us the map from $a \mapsto \hat{a}$, is norm decreasing (follows from Beurling theorem). To see the homomorphism explicitly we show the following:

$$(\Gamma(\lambda a + b))(\tau) = (\lambda a \hat{+} b)\tau = \tau(\lambda a + b) = \lambda\tau(a) + \tau(b) = \lambda\hat{a}\tau + \hat{b}\tau = (\lambda\Gamma(a) + \Gamma(b))(\tau)$$

$$\Gamma(ab)(\tau) = \hat{ab}\tau = \tau(ab) = \tau(a)\tau(b) = (\hat{a}\tau)(\hat{b}\tau) = (\Gamma(a)\tau)(\Gamma(b)\tau) = (\Gamma(a)\Gamma(b))\tau$$

□

9 Lecture 9: Clifford algebras and spin.

So now we know some fibre bundle theory, the overall aim is to construct the spin bundle, which is an associated vector bundle to the spin group. So we need to delve into what a spin group is. Which leads us to clifford algebras. The first concrete link between noncommutative geometry and the commutative world.

9.1 Clifford Algebras

Clifford algebras are a generalisation of the complex numbers. We can view the complex numbers and geometrical operations of the plane. With complex conjugate being the reflection about the x axis and rotations by multiplication of complex numbers with magnitude equal to one. Clifford algebras generalise this to higher dimensions and for this reason they are sometimes referred to as 'geometric algebras'. There construction seems a little contrived to begin with. But if you try to convert everything into geometrical operations it makes the reasons for certain choices a bit more apparent.

Definition 9.1: Bilinear form

Let V be a finite-dimensional vector space over a field $\mathbb{R}$. A function $\mathcal{B}\colon V \times V \to \mathbb{R}$ is a bilinear form if it satisfies the following conditions:

$$\mathcal{B}(x,y) = \mathcal{B}(y,x) \tag{11}$$
$$\mathcal{B}(\lambda x + y, z) = \lambda\mathcal{B}(x,z) + \mathcal{B}(y,z) \tag{12}$$
$$\mathcal{B}(x,u) = 0\ \forall u \in V \implies x = 0 \tag{13}$$

From a bilinear for we can construct a quadratic form Q, by setting $Q(x) = (x,x)$.

We can also go backwards from a quadratic form to a bilinear form by a process called polarisation. This is where $\mathcal{B}(x,y) = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right)$. A key observation of a bilinear form over a vector space is it's signature. This is found by picking an orthonormal basis of the vector space with respect to the form, $\{e_i\}_{i=1}^{n}$. Where n is the dimension of the vector space, and then order the $e_i's$ in such a way so that the first p all have norm $+1$ and the last q have norm -1, such that $n = p + q$. The signature of the space is then given by $s = q - p$.

Example 14. *The simplest example of a bilinear space is that of Euclidean space $\mathbb{R}^n$, which consists of the n-tuples of real numbers $(x_1, \ldots, x_n)$ together with the inner product*

$$\mathcal{B}(x,y) = \sum_{i=1}^{n} x_i y_i$$

.

Example 15. *Another familiar example of a bilinear space is that of Minkowski space $\mathbb{R}^(3,1)$, which consists of the 4-tuples of real numbers $(x_0, \ldots, x_3)$ together with the inner product*

$$\mathcal{B}(x,y) = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i$$

.

This can in fact be extended to arbitrary signature and dimension, denoted $\mathbb{R}^{p,q}$, with product

$$\mathcal{B}(x,y) = \sum_{i=1}^{p} x_i y_i - \sum_{j=1}^{q} x_j y_j$$

.

We are now ready to define a Clifford algebra over a vector space V.

Definition 9.2: Clifford algebra

A Clifford algebra over the vector space V with quadratic form Q, is denoted by $\mathcal{C}l(V,Q)$ and is the algebra generated by the product defined as $x \cdot x = Q(x)$.

If we take an orthonormal basis of V, we can express a familiar from physics relation between their products:

$$e_i e_j + e_j e_i = 2\mathcal{B}(e_i, e_j).$$

We can then extend this basis of V to a basis of $\mathcal{C}l(V)$ by taking all the products of the vector space basis elements and reordering them using the relation above.

9.2 Modules over Clifford algebras

A module over an algebra is the generalisation of a vector space over a field. However, the main differences are that algebras need not be commutative, so left and right actions are in general distinct. Also there may exist zero divisors ($xy = 0$ wit $x, y \neq 0$), which causes linear independent sets to be problematic to define. We will look at Clifford algebras over the vector space $\mathbb{R}^{p,q}$ with the usual bilinear form (inner product). We will denote this Cliffor algebra as $\mathcal{C}l_{p,q}$.

Definition 9.3: Clifford module

A left module over the Clifford algebra $\mathcal{C}l_{p,q}$, is a vector space V (over $\mathbb{R}$) together with an algebra morphism $L\colon \mathcal{C}l_{p,q} \to End(V)$. I.e for each element $a \in \mathcal{C}l_{p,q}$, there exists a linear transformation $L(a)$ of V such that:

$$L(ab + c) = L(a)L(b) + L(c)$$

$\forall a, b, c \in V$. The operator L is called left multiplication. A similar definition exists for a right module with a right multiplication, but now $R(ab) = R(b)R(a)$. A vector space V, that has both a left and right multiplication, so it is a left and right module, is called a $\mathcal{C}l_{p,q}$-bimodule.

9.2.1 Pin and Spin

To tidy up the next section, we will introduce some notation. There are some operations one can do on a Clifford algebra which have a geometrical meaning.

Definition 9.4

Given $\mathcal{C}l(V)$, let $a, b \in \mathcal{C}l(V)$ and $x \in V$. We can then define the following operations

- The main *antiautomorphism* is denoted $\bar{a}$ and is such that $\bar{x} = -x$, and $\bar{ab} = \bar{b}\bar{a}$.
- The operation called *reversion* is denoted a^* such that $x^* = x$ and $(ab)* = b^* a^*$
- the *main automorphism* is denoted a' and is such that $x' = -x$ and $(ab)' = a'b'$.

Now, a useful thing to know is that we can decompose a Clifford algebra into *even* and *odd* parts. We do this by using the main automorphism. Any element $a \in \mathcal{C}l(V)$ that maps to $+a$ is said to lie in the even part of the Clifford algebra and any element which maps to $-a$ is said to lie in the odd part of the Clifford algebra. Thus we can write the Clifford algebra as follows $\mathcal{C}l(V) = \mathcal{C}l^+(V) \oplus \mathcal{C}l^-(V)$.

Now we can define the Pin and Spin groups.

Definition 9.5: Pin and spin

The Pin group of (V, Q), denoted $Pin(V)$ is the subgroup of $\mathcal{C}l(V)$ generated by all elements $v \in V$ such that $Q(v) = \pm 1$. The Spin group is the intersection of $Pin(V)$ and $\mathcal{C}l^+(V)$.

Not very enlightening so we will explain next time how they related to rotations.